Weyl and the mathematisation of Quantum Mechanics

individual and collective perspectives.

Christophe Eckes

University Paris VII Denis-Diderot

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Introduction

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  - (ii) It becomes a journal as such in 1925.
- Most of the leading papers in atomic physics and quantum mechanics are published in ZfP, (articles by Heisenberg, Born, Jordan, Pauli, von Neumann, Wigner, etc.)
  - A significant exception : Schrödinger's great papers on the so-called wave mechanics appear in 1926 in the *Annalen der Physik*.

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- « the decline in the number of relativity publications had begun in 1924 in absolute numbers, but already in 1923 from a relative point of view, i.e., *before* the introduction of the « new » quantum theory in the following year. Indeed the real increase in quantum articles occurred only in 1927 ».

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(3) WEYL ascribes immediately a central role to group-theoretical methods in order to formalize quantum mechanics. He is not the first to do this :

- already in 1926, HEISENBERG uses the symmetric group of n elements $\mathfrak{S}_n$ to describe a quantum system consisting of n equivalent individuals.
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  (i) clarify the foundations of quantum mechanics,
  (ii) explain some qualitative experiments in spectroscopy.

B. *Gruppentheorie und Quantenmechanik*, i.e. Weyl's great monograph on quantum mechanics (1928)

- It derives from a lecture course given by WEYL at the ETH (Zürich) during the winter semester 1927-1928.
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- V « The symmetric permutation group and the algebra of symmetric transformations » (mathematics and physics).

Gruppentheorie und Quantenmechanik (GQ) and *Raum, Zeit, Materie* (1918-1923) have the same structure : mathematical and physical parts which are well delimited. They are addressed to MATHEMATICIANS and PHYSICISTS.

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- group-theoretical methods can be applied to *non relativistic* and *relativistic* quantum mechanics (DIRAC).
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WEYL's monograph also contains significant developments in the theory of group representations, independently from its application to quantum mechanics.

C. Some historical data

Until 1930, WEYL is professor of mathematics at the *ETH* (Zürich).

In late 1927, he announces a lecture course on group theory for winter semester 1927-1928. In september 1927, the two theoretical physicists in Zürich DEBYE and SCHRÖDINGER accept calls in other places.

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Until 1930, WEYL is professor of mathematics at the *ETH* (Zürich).

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- SCHRÖDINGER gives up is chair at the university of Zürich and goes to Berlin,
- DEBIE leaves the ETH on occasion of a call to Leipzig.

WEYL uses this opportunity to reorient his lecture course which is now devoted to group theory *and* quantum mechanics.

During the academic year 1928-1929, he holds a professorship in mathematical physics in Princeton University. He also gives lectures at the university of Berkeley.

The lecture course at Princeton has a great influence on the second edition of *Gruppentheorie und Quantenmechanik* (1931).

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More generally, WEYL's books derive from lecture courses :

- *Die Idee der Riemannschen Fläche* (1913),
- *Raum, Zeit, Materie* (1918-1923),
- *Gruppentheorie und Quantenmechanik* (1st and 2nd edition),
- *Classical groups* (1939).

D. Weyl : an isolated actor in the development of quantum mechanics ?

(a) WEYL refers to many sources in this monograph :

- SCHRÖDINGER's wave mechanics,
- HEISENBERG-BORN-JORDAN's matrix mechanics,
- VON NEUMANN's implicit definition of an ABSTRACT Hilbert space,
- The spin hypothesis and its implications (GOUDSMIT, UHLENBECK, PAULI),
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GQ is the first monograph containing VON NEUMANN's axiomatization of Hilbert spaces and it reflects the last refinements of quantum mechanics.

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   - In « Quantisierung als Eigenwertproblem » (Part I), SCHRÖDINGER mentions WEYL who helped him to formulate the time-independent Schrödinger equation in spherical coordinates.

(ii) In 1925, WEYL is in contact with JORDAN and BORN concerning the foundations of matrix mechanics,

(iii) There is an important correspondence between WEYL and VON NEUMANN in 1925-1930,

(iv) WEYL is also in correspondence with DIRAC, HEISENBERG, etc.

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E. Elias : a key-reference in our presentation

To this end, we will use the conceptual framework due to the historian and sociologist N. ELIAS (1897-1990), — cf. *The society of individuals* (1939).

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A basic reference to explain the connections between individual trajectories and collective processes. Three crucial issues :

- the characterization of collective processes,
- the situation of individuals within collective processes,
- the description of the *social structures* which influence *individual choices*.

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Conversely, group-theoretical methods are shared by different scientists in quantum mechanics : HEISENBERG, WIGNER, VON NEUMANN, WEYL, LONDON, HEITLER, VAN DER WAERDEN etc. The problem consists in

(i) describing this collective process,
(ii) locating WEYL within it,
(iii) explaining its impact on a broader public of physicists involved in the development of quantum mechanics.

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F. Plan of our presentation :

FIRST PART : a brief comment on *The society of individuals* (ELIAS).

- The concept of « reciprocal relationships »,
- « individuals » within « collective processes »,
- application to our object of study.

SECOND PART : partial description of the network which determine the genesis of *Gruppentheorie und Quantenmechanik*.

- WEYL (1925-1926) at the intersection of two projects : matrix mechanics / wave mechanics,
- a crucial relationship with VON NEUMANN.

SECOND PART : partial description of the network which determine the genesis of *Gruppentheorie und Quantenmechanik*.

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THIRD PART : group-theoretical methods in quantum mechanics, a collective project.

- a project shared by german-speaking scientists,
- *Gruppentheorie und Quantenmechanik* as a broad synthesis on group theory and its application to quantum mechanics,
- The central role of WEYL to justify group-theoretical methods in quantum mechanics,
- A contrasted reception among (theoretical) physicists.

Elias' conceptual and theoretical framework

It may be surprising to refer to ELIAS : his work doesn't belong to history or sociology of science. Nevertheless his concepts are very useful to describe the connections between individual trajectories and collective processes in general and particularly in history of science.

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(2) In this last work, he criticizes simultaneously two opposite points of view concerning the connection between « individuals » and « society » : on the one hand the so-called METHODOLOGICAL INDIVIDUALISM, on the other hand the so-called HOLISM.

I.1. Elias' objections against methodological individualism

« Individuals » are considered as basic elements of a society. We could describe individuals almost independently from the social structures, the institutions, etc. they belong to.

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ELIAS underlines the naivety of this opinion :

- A social structure doesn't consist merely of « individuals ».
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Accordingly, Weyl's interest in quantum mechanics doesn't depend merely on individual choices or internal changes within his work but on a series of reciprocal relationships with Schrödinger, Born, Jordan, von Neumann, etc. in 1925-1931.

According to methodological individualism, a collective process is just a sum of individual decisions. On the contrary, ELIAS shows that a collective process « is more and other than a collection of separate individuals » (p. 7).

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For instance, let us admit that the project of using group-theoretical methods in quantum mechanics is a collective process.

- Then it is not sufficient to describe separately the works of HEISENBERG, WEYL, VON NEUMANN etc. in order to grasp this process.
- Furthermore, we will have to locate each of these actors within this process.
- For instance, we will show that WEYL plays a central role by describing his relationships with scientists sharing the same methods.

I.2. The antinomy between methodological individualism and holism

In a HOLISTIC perspective

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However, his mathematical practice and his work are characterized by a tension between different traditions depending on distinct institutions (for instance the university of Göttingen and the ETH Zürich).

- Holism is not the right « level » to explain this tension.

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He assumes the existence of individual decisions : individuals are not merely determined by collective processes (vs holism). But an individual choice is always limited by a given social structure (vs methodological individualism).

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He assumes the existence of individual decisions : individuals are not merely determined by collective processes (vs HOLISM). But an individual choice is always limited by a given social structure (vs METHODOLOGICAL INDIVIDUALISM).

« Every large and complex society has, in fact, both qualities : it is very firm and very elastic. Within it scope for individual decision constantly appears. Opportunities present themselves that can be either seized or missed. Crossroads appear at which people must choose, and on their choices, depending on their social position, may depend either their immediate personal fate or that of a whole family (...). But the opportunities between which a person has to choose in this manner are not themselves created by that person. They are prescribed and limited by the specific structure of his society and the nature of the functions the people exercise within it ».

Moreover, it is naïve and contradictory to believe that individual decisions could radically change the shape of a social structure :

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« No individual person, no matter how great his stature, how powerful his will, how penetrating his intelligence, can breach the autonomous laws of the human network from which his actions arise and into which they are directed ».

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- in a series of lecture courses and / or conferences given in Zürich, Göttingen, Hambourg, Princeton, Berkeley etc.

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Nevertheless, he will have a very limited impact on physicists. Group theory won't play suddenly a central role in the formalization of quantum mechanics at the end of the 20's.

Let us explain how ELIAS overcomes the antinomy between METHODOLOGICAL INDIVIDUALISM and HOLISM :

He focuses on the webs of reciprocal relationships between individuals. These « webs » can be viewed as a middle term between

- individuals,
- social structures.

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The alternative between METHODOLOGICAL INDIVIDUALISM and HOLISM is artificial. Social networks are an essential tool in order to describe the connections between individuals and social structures.

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- localization of individuals belonging to it (Do they have a central position or not within a given network ?)
- One individual can be simultaneously involved in different networks (this is particularly true with von Neumann in quantum mechanics)

In the next part of our presentation, we will show that the first edition of *GQ* reflects (reciprocal) relationships between Weyl and other scientists, mainly Schrödinger, Heisenberg, Born / Jordan, von Neumann.

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In the third part of our presentation, we will describe the network of mathematicians and physicists sharing group-theoretical methods in quantum mechanics during the period 1926-1931. We will have to locate WEYL's works within this little network.

Part II. : *Gruppentheorie und Quantenmechanik* : a wide synthesis in quantum mechanics

II.1. Weyl's involvement in mathematical physics

WEYL's research in mathematical physics begins in 1916 / 1917. Between 1917 and 1921, he works on general relativity and on a project of unified fields theory.

He is professor at the ETH from the fall 1913 until 1930. Nevertheless, he is deeply influenced by the Göttingen school of mathematical physics in 1917-1921 :

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- MIE-HILBERT programm = description of the structure of matter within a (classical) field theory,
- pre-established harmony between mathematics and physics (MINKOWSKI, HILBERT),
- variational methods which are praised by HILBERT.

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WEYL develops a unified fields theory between 1918 and 1921.

- it is based on a « purely infinitesimal geometry », i.e. a generalization of Riemannian geometry,
- it can be considered as a first gauge theory (roughly speaking a field theory in which the lagrangian is invariant under a continuous group of local transformations).

WEYL is convinced that a purely *a priori* geometrical framework can lead to a consistent physical theory. (pre-established harmony between mathematics and physics).

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WEYL's theory is disproved by theoretical physicists (mainly EINSTEIN), philosophers (REICHENBACH) and mathematicians (HILBERT).

Similarities between *Raum, Zeit, Materie* and *Gruppentheorie und Quantenmechanik* :

- the same implicit readers : mathematicians and physicists,
- the same presentation : mathematical theories are first required before studying respectively relativity theories and quantum mechanics,
- in these two monographs, WEYL shows great interest in the unification of physics and physical interactions.

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- in these two monographs, WEYL shows great interest in the unification of physics and physical interactions.

Differences :

(1) Connection between mathematics and physics.

- In 1918-1921, theoretical physicists consider him as a foolhardy mathematician because of his unified fields theory.
- In *GQ*, Weyl admits that he is just a mathematician. He must base his reflections on results belonging to empirical and theoretical physics.

(2) Reception by physicists.

- Roughly speaking, *Raum, Zeit, Materie* has immediately a great success among physicists, although they perceive WEYL's gauge theory as a mathematical speculation which lacks empirical foundations.
- On the contrary, *Gruppentheorie und Quantenmechanik* seems very difficult to physicists because they are not prepared to understand group theory (in a wide sense) and its application to quantum mechanics.

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(3) The mathematical framework.

- Differential and Riemannian geometry, tensor calculus in *Raum, Zeit, Materie*,
- Functional analysis, representations of finite groups and Lie groups in *Gruppentheorie und Quantenmechanik*.

II.2. reciprocal relationships with other physicists

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Although he is working on Lie groups and on a book on philosophy of mathematics in 1925-1926, WEYL is taking a keen interest in the foundations of quantum mechanics.

E. SCHOLZ : « [WEYL] was well aware what was going on in quantum mechanics. Even more than that, he actively participated in the internal discourse of the protagonists. He was in regular communication with E. SCHRÖDINGER who taught at the university of Zürich in direct neighborhood to the ETH where WEYL was teaching. And he continued to be a kind of external « corresponding member » of the Göttingen mathematical science milieu ».

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- In 1925, Born and Jordan clarify Heisenberg's research in their article : « Zur Quantenmechanik »,
- Immediately after the publication of this paper, they keep up a correspondence with Weyl.

In the second chapter of *GQ*, Weyl combines Schrödinger's wave mechanics and Heisenberg-Born-Jordan's matrix mechanics.

His ability to understand these two projects is not surprising if we indicate that he is already in communication with all these protagonists in 1925-1926.

II.3. Schrödinger's wave mechanics.

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« Wave mechanics » consists in expanding the duality wave / particle to matter.

(i) Schrödinger and de Broglie share a realistic conception of matter waves.
(ii) They consider the historical development of modern physics in the same way.
    - « Wave mechanics » is « the » missing piece in this history.

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Moreover, he solves a well-known kind of problems in physics, a so-called « Eigenwertproblem » which corresponds to the resolution of the time-independent Schrödinger equation

$$H\psi = E\psi$$

where (the hamiltonian) H is a second-order differential operator and ψ the wave function, i.e. an eigenfunction of H corresponding to the eigenvalue E (the energy).

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In a modernized way and more generally, an « Eigenwertproblem » consists in determining the spectrum of eigenvalues of a self-adjoint operator acting on a (complex) Hilbert space.

In *Methoden der mathematischen Physik* (1924), COURANT and HILBERT attach great importance to the resolution of several « Eigenwertprobleme ».

- See for instance chapter V of this book which is entitled : « Die Schwingungs- und Eigenwertprobleme der mathematischen Physik ».

In *Methoden der mathematischen Physik* (1924), Courant and Hilbert attach great importance to the resolution of several « Eigenwertprobleme ».

- See for instance chapter V of this book which is entitled : « Die Schwingungs- und Eigenwertprobleme der mathematischen Physik ».

To sum up

- Schrödinger bases his approach on a physical intuition (the matter waves),
- he refers to physical theories which are well-constituted,
- he uses mathematical tools which are commonly shared by physicists.

WEYL is aware that « wave mechanics » is suitable to physicists. That's why the second chapter of *GQ* begins with DE BROGLIE's and SCHRÖDINGER's approach :

Weyl is aware that « wave mechanics » is suitable to physicists. That's why the second chapter of *GQ* begins with de Broglie's and Schrödinger's approach :

« This approach seems to me less cogent, but it leads more quickly to the fundamental principles of quantum mechanics and to the most important consequences of experimental science. We shall therefore follow it, since we are more concerned in giving a short but comprehensive account than in giving a complete discussion of the physical foundations ». [GQ, p. 48]

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(1) WEYL describes « DE BROGLIE waves of particles » (II, §2),
(2) he sums up SCHRÖDINGER's analogy : geometrical optics / wave optics, hamiltonian mechanics / wave mechanics (II, §3),
(3) he formulates the time-independent SCHRÖDINGER wave equation.

For a single particle with potential energy V, this equation takes the form

$$\frac{\hbar}{2m}\nabla^2\psi + [E - V(xyz)]\psi = 0,$$

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WEYL interprets the « Eigenwertproblem » corresponding to the resolution of this equation as follows :

« The problem is thus reduced to finding values of E and functions $\psi \neq 0$ of position which satisfy this equation and are such that the integral of $\psi\overline{\psi}$ over the entire space is finite. They are the characteristic numbers and characteristic vectors of the Hermitian [self-adjoint] operator H associated with the energy in the function space of all functions of position ψ. *The characteristic numbers E are the possible energy levels of the particles* ». *[GQ p. 56]*

II.4. Matrix mechanics and the « Göttingen milieu » in physics.

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Heisenberg, Born and Jordan formulate explicitly the foundations of matrix mechanics in a series of three articles :

- « Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen » (Heisenberg, *ZfP*, **33**)
- « Zur Quantenmechanik » (Born, Jordan, *ZfP*, **34**)
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In their paper « Zur Quantenmechanik », BORN and JORDAN interpret HEISENBERG's results in the framework of matrix calculus.

In his Nobel lecture (1954), BORN tells us the following story :

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« HEISENBERG banished the picture of electron orbits with definite radii and periods of rotation because these quantities are not observable, and insisted that the theory be built up by means of (...) square arrays (...). Instead of describing the motion by giving a coordinate as a function of time, $x(t)$, an array of transition amplitudes x_{mn} should be determined. To me the decisive part of his work is the demand to determine a rule by which for a given array the array of the square can be found (or, more general, the *multiplication rules* for such arrays). (...) After a week of intensive thought and trial I suddenly remembered an algebraic theory which I had learned from my teacher, Professor ROSANES, in Breslau. Such square arrays are well known to mathematicians and, in conjunction with a specific rule for multiplication, are called matrices ».

« (...) and at once there stood before me the peculiar formula

$$pq - qp = h/2\pi i.$$

This meant that coordinates q and momenta p cannot be represented by figure values but by symbols, the product of which depends upon the order of multiplication — they are said to be "non-commuting" ».

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According to BORN, a simple memory of lecture courses he attended in Breslau could explain his main discovery : HEISENBERG's multiplication rule corresponds to multiplication of square matrices.

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- M. Bôcher, *Introduction to Higher Algebra* (tr. in German in 1910). In this book, Bôcher uses the axiomatic method in order to describe the rules of matrix calculus and its properties.
- D. Hilbert and R. Courant, *Methoden der mathematischen Physik* (1924). They consider matrix calculus as a possible tool in the formalization of physical theories.

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- D. HILBERT and R. COURANT, *Methoden der mathematischen Physik* (1924). They consider matrix calculus as a possible tool in the formalization of physical theories.

WEYL also mentions these books in *GQ*. WEYL and BORN-JORDAN describe matrix calculus in the same way.

Contrary to wave mechanics, matrix mechanics

(i) is not based on a physical intuition, the only rule followed by HEISENBERG consists in avoiding non observable quantities ;

(ii) is considered by HEISENBERG, BORN and JORDAN as a break in the historical development of physics,

- In his inaugural article HEISENBERG compares classical mechanics to quantum mechanics in order to describe the gap separating them ;

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Because of these three arguments, WEYL begins his chapter on the foundations of quantum mechanics with DE BROGLIE's and SCHRÖDINGER's wave mechanics.

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(i) In the resolution of the so-called « space problem » (1921-1923), he practices intensively matrix calculus and decomposition of matrices ;

(ii) this mathematical framework is very useful in order to build up a consistent theory in quantum mechanics,

- it confirms the importance of mathematical physics in the development of a physical theory ;

(iii) WEYL shows great interest in BORN-JORDAN's formula relating impulsion to position (commutation law).

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More precisely, WEYL aims at clarifying the mathematical foundations of this formula (cf. letters to BORN and JORDAN (1925)). To this end, WEYL uses some results which belong to the theory of Lie groups and Lie algebras

E. Scholz : « Weyl was well informed about the work done by the Göttingen physicists and even contributed actively to the research discussion among Born, Jordan and Heisenberg in the crucial months of mid and late 1925. In September 1925, Born visited Weyl at Zürich and reported him about the latest progress in quantum mechanics ».

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In a letter to Born (September 27, 1925), Weyl explains the importance of group-theoretical methods in order to deduce Born-Jordan's formula.

- In his answer Born underlines that Weyl's method « is difficult for physicists to access »,
- moreover, according to Born and Jordan (who also read Weyl's letter to Born), this formula must be considered as an independent assumption which doesn't need further mathematical justification.

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Cf. Interview of HEISENBERG by KUHN on 19 February 1963. During the conversation, HEISENBERG mentions NOETHER's paper entitled « Invariante Variationsprobleme » (1918) which consists in relating conservation laws to infinitesimal generators of a Lie group :

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« Much later, of course, the physicists recognized that the conservation laws and the group theoretical properties were the same. (...) But at that time [in 1925], this connection was not so clear. Well, it was apparently clear to NOETHER, but not for the average physicist. Also in Göttingen it was not clear. The NOETHER paper has been written in Göttingen, I understand. But it was not popular among the physicists, so I certainly wouldn't learn that from BORN in Göttingen ».

« I'm sure that the paper itself did not play a large role for the development of quantum theory. It did play a role for the development of general relativity. It was actually formulated in connection with general relativity, which was an interest with (Hilbert's) group and therefore also Noether. But it did not penetrate into the circles of quantum theory, so I didn't realize the importance of that paper ».

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Scientists who apply group-theoretical methods to quantum mechanics have a close connection with Zürich and Göttingen.

But it doesn't imply that these methods suddenly play a central role in the development of quantum mechanics, *even at Göttingen*.

- In 1925, BORN and JORDAN are sceptical about this approach (letters to WEYL).
- Moreover, at the end of the 20's BORN will reject group theory in the formalization of quantum mechanics.

II.5. Weyl and von Neumann

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Roughly speaking, an abstract (complex) Hilbert space is a complex inner product space that is complete (i.e. it satisfies Cauchy criterion of completeness).

(1) a complex PRE-HILBERT SPACE is a complex vector space in which there is an inner product $\langle x, y\rangle$ satisfying the following properties

- $\langle y, x\rangle = \overline{\langle x, y\rangle}$, for all $x,\ y \in H$,
- $\langle ax + by, z\rangle = a\langle x, z\rangle + \underline{b}\langle y, z\rangle$, for all $x, y, z \in H,\ a, b \in \mathbb{C}$
- $\langle x, ay + bz\rangle = \overline{a}\langle x, y\rangle + \overline{b}\langle x, z\rangle$, for all $x, y, z \in H,\ a, b \in \mathbb{C}$
- $\langle x, x\rangle \geq 0$

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VON NEUMANN and WEYL restrict themselves to *separable* Hilbert spaces, i.e. Hilbert spaces that admit a countable orthonormal basis.

von Neumann, « Mathematische Begründung der Quanten-mechanik » (*Nachrichten*, Göt., 1927).

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(ii) then he associates matrix mechanics with ℓ^2, i.e. the space of infinite sequences of complex numbers $a = \{a_i\}$ satisfying the following property : $\sum_{i=1}^{\infty} |a_i|^2 < +\infty$

- inner product on ℓ^2 : $\langle a, b\rangle = \sum_{i=1}^{\infty} a_n \overline{b_n}$.

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(iii) he associates wave mechanics with $L^2(\mathbb{R}^3)$, i.e. the space of square-integrable functions defined on $\mathbb{R}^3$. A function f on $\mathbb{R}^3$ is square integrable if $\int_{R^3} |f(x)|^2 \mathrm{d}x < +\infty$.

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(v) he finally defines an (abstract) complex separable Hilbert space.

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(1) A General argument concerning the community of mathematicians linked with Göttingen, cf. interview of Heisenberg by Kuhn :
H — Hilbert had asked me to give talks to the mathematical section of the faculty (...). So I gave a number of talks and then I wrote the paper for the mathematicians. Hilbert found it was so important that it should be presented in a way which appealed to the mathematicians.

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H — HILBERT had asked me to give talks to the mathematical section of the faculty (...). So I gave a number of talks and then I wrote the paper for the mathematicians. HILBERT found it was so important that it should be presented in a way which appealed to the mathematicians. K — Were the mathematicians in fact quite interested ? (...)
H — Well, at this stage they did, yes. And there was some excitement about it. (...) I think Courant was interested. (...) Of course, COURANT was completely interested later on when he saw the SCHRÖDINGER picture come in. Then, of course, there was the problem of the Hilbert space which was exciting for the mathematicians.

Let us recall that von Neumann has a close connection with the Göttingen milieu.

- he regularly visits Göttingen in 1927-1928,
- Nordheim, Hilbert and von Neumann publish an article on the axiomatization of Quantum mechanics (1928),
- His works on Hilbert spaces have a great impact on mathematicians at Göttingen.
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Furthermore, there exists an important correspondence between Weyl and von Neumann within the period 1925-1930

In their correspondence Weyl and von Neumann have two main subjects :

(i) von Neumann's research in functional analysis (more precisely operator theory),
  - In the 2nd edition of *GQ*, Weyl mentions von Neumann's works on « unbounded operators » — an unbounded operator on a Hilbert space H is a linear operator whose domain is a linear subspace of H.
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   - WEYL, *GQ*, p. 40 : « J. VON NEUMANN has gone furthest in dealing with linear operators for which boundedness is not postulated ».

(ii) WEYL's monograph in quantum mechanics
   - we learn from this correspondence that VON NEUMANN receives the proofs of *GQ* in summer 1928. He suggests WEYL some improvement.

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In other words, VON NEUMANN's article is in accordance with WEYL's way of defining and using mathematical concepts.

Because of all these arguments, it seems less surprising to find a summary of VON NEUMANN's research on Hilbert spaces in WEYL's monograph.

Because of all these arguments, it seems less surprising to find a summary of von Neumann's research on Hilbert spaces in Weyl's monograph.

Weyl, I, § 7 : « The unitary spaces which appear in quantum mechanics usually have *an infinite number of dimensions*. Such a space consists of all vectors

$$\mathfrak{x} = (x_1, x_2, ...)$$

whose components x_i constitute an infinite sequence of numbers for which

$$\mathfrak{x}^2 = \overline{x}_1 x_1 + \overline{x}_2 x_2 + ...$$

converges. Within this domain addition and multiplication with numbers, as well as the construction of the scalar product of two vectors, are possible. All the axioms employed so far are satisfied, with the exception of the dimensionality axiom γ (...). Since the vector components x_1, x_2, ... constitute a denumerable set, this "Hilbert space" has a denumerably infinite number of dimensions ».

group-theoretical methods in quantum mechanics : a collective project

III.1. a little but a dense network

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- Heisenberg (1926)
- Wigner (1926-1931)
- von Neumann (1927-1928)
- Weyl (1927-1931)
- van der Waerden (1929-1931)
- Heitler (1927-1928)
- London (1927-1928)

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SCHOLZ : « Walter HEITLER and Fritz LONDON had come to Zürich on Rockfeller grants in 1926 (F. LONDON), respectively 1927 (W. HEITLER), to work with E. SCHRÖDINGER ».

« In summer 1927, E. Schrödinger went from Zürich to Berlin, as a successor on M. Planck's chair ; in October F. London joined him there as an assistant. W. Heitler, whose Rockfeller grant had run out more or less at the same time, accepted an offer from Max Born to become an assistant at Göttingen. There he got to know E. Wigner whose group theoretic works he had started to read with great interest when in Zürich ».

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All these protagonists form a little network which is very dense :

- reception of HEISENBERG article (1926) by WIGNER,
- reception of WEYL's article on Lie groups (1925-1926) by VON NEUMANN and WIGNER (1927-1928),
- VON NEUMANN and WIGNER work together at Göttingen,
- HEITLER and LONDON work together at Zürich,
- connection between HEITLER and WIGNER at Göttingen.

Situation of WEYL within this network ?

Situation of Weyl within this network ?

- correspondence with Heisenberg and von Neumann,
- Weyl mentions Wigner's and von Neumann's articles on the theory of group representations and its application to quantum mechanics,
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- and later correspondence with van der Waerden.

Weyl plays a central role in this collective project. Moreover, his monograph contains all the known applications of group theory to quantum mechanics. This is in accordance with his central position.

III.2. which part of group theory do they use ?

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(1) His main reference in algebra is SERRET's *Cours d'algèbre supérieure* (third edition, 1866, german translation in 1868)

(2) In other words, he doesn't mention more recent algebraic texts, for instance
   - H. WEBER, *Lehrbuch der Algebra* (1895-1896)
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(3) Heisenberg's mathematical framework is the *algebraic equation theory* and not *the representation theory of finite groups*.

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- At the beginning of his second paper, he explicitly refers to FROBENIUS, SCHUR and BURNSIDE.
- He justifies the use of this mathematical tool as follows :
  - it doesn't lead to contradictions,
  - it doesn't imply complex and long calculation.

A *linear representation* of a finite group on a finite-dimensional complex vector space V is a group homomorphism $\rho : G \longrightarrow \mathrm{GL}(V)$, that is

- $\rho(e) = Id_V$ where e denotes the identity of G,
- $\rho(g^{-1}) = \rho(g)^{-1}$, for all g in G,
- $\rho(gg') = \rho(g)\rho(g')$ for all g, g' in G.

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A group representation is said to have the *complete reducibility property* if it can be decomposed into a direct sum of irreducible sub-representations.

Definition of the *regular representation* of a finite group : let G be a finite group of order g and let V be a vector space of dim g with a basis $(e_t)_{t \in G}$ indexed by the elements t of G,
For $s \in G$, let ρ_s be the linear map of V into V which sends e_t to e_{st} ; this defines a linear representation which is called the (left) regular representation of G.

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NB Wigner refers to Speiser's monograph on finite groups and their representations (1923). This book popularizes the theory of group representations among (german-speaking) mathematicians.

Quantum mechanics requires all the aspects of this mathematical theory :

(1) representations of finite groups / Lie groups (Lie group = set endoved simultaneously with the compatible structures of a group and a $\mathscr{C}^\infty$ manifold.)

- For instance, the study of the linear representations of $\mathrm{SO}(3)$ plays a central role in (non relativistic) quantum mechanics.

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Conversely, the theory of group representations has all kinds of applications in quantum mechanics. In particular, it is possible to extend this mathematical framework to quantum systems of increasing complexity.

III.3. various applications of group theory to quantum mechanics

According to WEYL, theory of group is essential in the foundations of quantum mechanics :

- « it has recently been recognized that group theory is of fundamental importance for quantum physics ; it here reveals the essential features which are not contingent on a special form of the dynamical laws nor on special assumptions concerning the forces involved ».

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- « it has recently been recognized that group theory is of fundamental importance for quantum physics ; it here reveals the essential features which are not contingent on a special form of the dynamical laws nor on special assumptions concerning the forces involved ».

and in the formalization of empirical data resulting from spectroscopy :

- « The investigation of group first becomes a connected and complete theory in *the theory of the representation of groups by linear transformations*, and it is exactly this mathematically most important part which is necessary for an adequate description of the quantum mechanical relations ».

Group theory is used in four areas during the period 1926-1931 (cf. M. SCHNEIDER) :

(1) « Foundational questions » (WIGNER, WEYL)
   - for instance, WIGNER identifies the « conservation laws » of quantum systems by using group-theoretical methods.

Group theory is used in four areas during the period 1926-1931 (cf. M. Schneider) :

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(3) « Molecular bond » (Heitler, London, Weyl)
(4) « Dirac wave equation », extension to relativistic quantum mechanics (Weyl, van der Waerden)

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« WIGNER and WEYL not only introduced group representations into quantum mechanics in quite different ways with different goals but they reached this interaction between physics and mathematics from opposite directions. While WIGNER was above all a theoretical physicist, WEYL was a pure mathematician ».

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Moreover, in 1927 VON NEUMANN advises WIGNER to read WEYL's article on Lie groups — essentially the second part which is devoted to $SO(n)$ and its representations. WIGNER and VON NEUMANN use this reference in a series of papers entitled « Zur Erklärung einiger Eigenschaften der Spektren aus der Quantenmechanik des Drehelektrons ».

(2) There are unexpected differences between van der Waerden and Weyl :

- In a conference entitled « Topologie und abstrakte Algebra als zwei Wege des mathematischen Verständnisses » (1931), Weyl criticizes abstract algebra : this mathematical domain is too general, it can't lead by itself to « effective knowledge » in pure mathematics.
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However, VAN DER WAERDEN and WEYL do not disagree on the role of abstract algebra in the formalization of quantum mechanics.

Indeed, WEYL's and VAN DER WAERDEN's monographs on quantum mathematics « are both based on abstract algebra » [M. SCHNEIDER]. In particular, WEYL sets out the theory of abstract groups in the third chapter of *GQ*.

WEYL's point of view on « abstract algebra » is all but clear at the beginning of the 30's.

- He recognizes that it plays a central role in the development of quantum mechanics,
- On the other hand, he criticizes with vehemence its hegemony in pure mathematics at Göttingen.

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Let us clarify now the main differences between WEYL and VAN DER WAERDEN in the field of quantum mechanics.

- WEYL's monograph is written for physicists *and* mathematicians. On the contrary, VAN DER WAERDEN is pragmatical : his book is first addressed to physicists.

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- WEYL's monograph is written for physicists *and* mathematicians. On the contrary, VAN DER WAERDEN is pragmatical : his book is first addressed to physicists.
- WEYL is convinced that group-theoretical methods are unavoidable in quantum mechanics, whereas VAN DER WAERDEN admits the relevance of group free methods (for instance SLATER's approach, based on « traditional algebraic tools » [SCHOLZ])

Let us sum up our arguments on WEYL :

- WEYL occupies a central position in the network of scientists using group-theoretical methods in quantum mechanics,
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- he dogmatically advocates for group-theoretical methods in mathematical physics (in his monograph, in his articles and also in his talks and lecture courses during his stay in the United States (mainly Princeton and Berkeley, 1929)).

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The other protagonists belonging to this network are not so intransigent.

III.4. Reception of these group-theoretical methods among physicists

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There is no antagonism between « two camps » : defenders vs detractors of group-theoretical methods in quantum mechanics. More precisely, it would be misleading to focus on two extremes :

- Weyl as the most intransigent advocate of these methods,
- Born and Slater as the most virulent detractors of group theory when applied to quantum mechanics.

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Among scientists using group-theoretical methods in quantum mechanics, some of them are conciliatory : they admit the relevance of other approaches (cf. van der Waerden).

Heitler is another interesting case. He becomes Born's assistant at the end of the 20's. Under Born's influence, he doesn't use anymore group-theoretical methods in the description of molecular bonds.

Conversely, theoretical physicists do not necessary agree with BORN's and SLATER's opinion, following which group-theoretical methods must be avoided in quantum mechanics because they are too technical. Three examples :

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(ii) In a 1929 letter to WEYL, SCHRÖDINGER claims that we must clarify first the physical foundations of quantum mechanics before using these methods (scepticism but not rejection).

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(iii) HEISENBERG is clearly enthousiastic in his recension of WEYL's monograph.

the reception of works using these methods in quantum mechanics is very complex and contrasted among physicists :

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- Rejection (BORN, SLATER),
- Interest (EHRENFEST, SOMMERFELD),
- Support (HEISENBERG, UHLENBECK, LAPORTE, CASIMIR),
- Scepticism (HARTREE, SCHRÖDINGER).

In particular, SCHNEIDER describes in detail the reception of WEYL's monograph by physicists. Such a reception is crucial in order to determine this typology.

It seems clear that theoretical physicists are not massively opposed to these new methods. Moreover the three monographs due respectively to WEYL (1928, 1931), WIGNER (1931) and VAN DER WAERDEN (1932) have a certain audience among physicists.

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SCHOLZ : « With the exception of such « heroic » but for a long time relatively isolated contributions, it needed a new generation of physicists and a diversification of problems and another problem shift in quantum physics, before group theory was stepwise integrated into the core of quantum physics ».